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# THE DYNAMICS OF A LAGRANGE TOP WITH A VIBRATING SUSPENSION POINT†

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The motion of a Lagrange top, whose suspension point performs high-frequency vertical harmonic oscillations of small amplitude, is considered. The angular velocities of the natural rotation of the top and of the rotation of its axis of symmetry around the vertical are assumed to be small. It is well known that, in the case of a classical Lagrange top with a fixed suspension point, for any values of the parameters of the problem (the values of the constants of cyclical integrals) there is a unique regular precession of the top. When the suspension point vibrates the following result is established, which has no analogues in the classical problem: regions are distinguished in the plane of those parameters in which, for any position of the centre of gravity of the top on the axis of symmetry, there is a unique periodic motion of the top (with a period equal to the period of oscillations of the suspension point), close to regular precession, and also regions in which, depending on the position of the centre of gravity, there can be one or three such motions. A rigorous solution of the problem of the stability of these motions of the top is given using the methods of the KAM theory. © 2000 Elsevier Science Ltd. All rights reserved.

This paper is a development of the results obtained in [1], where the problem of the periodic motions of a spherical pendulum with a vibrating suspension point is solved with assumptions similar to those used here.

A number of investigations have been devoted to different aspects of the problem of the dynamics of a rigid symmetrical body with a vibrating suspension point: the motion of a rapidly rotating symmetrical and close to symmetrical gyroscope when there are vertical vibrations of the suspension point has been investigated in [2, 3], the behaviour of a Lagrange top when the suspension point performs harmonic oscillations in a horizontal plane was considered in [4], the motion of a viscoelastic rigid body with a moving base was investigated in [5], and the rotation of a Lagrange top when there are random oscillations of the point of support was considered in [6].

## 1. FORMULATION OF THE PROBLEM. CONVERSION OF THE HAMILTON FUNCTION

Consider a dynamic symmetrical rigid body moving in a uniform gravity field around a fixed point $O$. Suppose the centre of mass of the body lies on its dynamic-symmetry axis. This rigid body is called a Lagrange top; its motion was investigated in detail in [7–9].

We will assume that the point $O$ executes vertical motion in accordance with the law $O_*O = \xi(T)$ about a certain fixed point $O$. Suppose $OXYZ$ is a system of coordinates moving translationally in absolute space (the $OZ$ axis is directed vertically upwards) and $Oxyz$ is a system of coordinates, rigidly attached to the body, whose axes coincide with the principal axes of inertia of the body for the point $O$, where the $Oz$ axis is directed along the dynamic-symmetry axis, and the centre of mass $G$ of the body lies on the positive semiaxis $Oz(OG = z_G, z_G > 0)$. We will specify the orientation of the system of coordinates $Oxyz$ with respect to $OXYZ$ using the Euler angles.

The kinetic energy of the body is given by the expression

$$T = \frac{1}{2}m\mathbf{v}_O^2 + m\mathbf{v}_O \cdot \mathbf{v}_{G_{\rm rel}} + \frac{1}{2}[A(p^2+q^2)+Cr^2] \tag{1.1}$$

where $m$ is the mass of the body, $A$ and $C$ are the equatorial and axial moments of inertia respectively, $\mathbf{v}_o$ is the velocity of the point $O$, $\mathbf{v}_{G\rm rel} = \boldsymbol{\omega} \times \overrightarrow{OG}$ is the velocity of the point $G$ in the system of coordinates $OXYZ$, and $\omega$ is the vector of the absolute angular velocity of rotation of the body, having projections $p$, $q$ and $r$ in the attached system of coordinates.

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In projections onto the $Oxyz$ axes we have $\overrightarrow{\mathbf{OG}} = (0, 0, z_G)^T$, $\mathbf{v}_{G\text{rel}} = (qz_G, -pz_G, 0)^T$, $\mathbf{v}_O = \dot{\xi}\mathbf{n}$, where $\mathbf{n} = (\sin\theta\sin\varphi, \sin\theta, \cos\varphi, \cos\theta)^T$ is the unit vector of the vertical axis $OZ$.

From (1.1) and Euler's kinematic equations we have the following expression for the kinetic energy of the body

$$T = \frac{1}{2}m\dot{\xi}^2 - mz_G\dot{\xi}\dot{\theta}\sin\theta + \frac{1}{2}A(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}C(\dot{\psi}\cos\theta + \dot{\varphi})^2 \tag{1.2}$$

The potential energy of the body can be calculated from the formula

$$\Pi = mgz_G\cos\theta + mg\xi(t) \tag{1.3}$$

It follows from (1.2) and (1.3) that the coordinates $\psi$ and $\varphi$ are cyclical, and the momenta corresponding to them are the same as in the case of the motion of a Lagrange top with a fixed point $O$

$$p_\psi = A\dot{\psi}\sin^2\theta + C(\dot{\psi}\cos\theta + \dot{\varphi})\cos\theta \tag{1.4}$$
$$p_\varphi = C(\dot{\psi}\cos\theta + \dot{\varphi})$$

We will introduce the notation $p_\psi = Aa, p_\varphi = Ab$ for the constant quantities $p_\psi$ and $p_\varphi$ (where $a$ and $b$ are constants). We then have from (1.4)

$$\dot{\psi} = \frac{a - b\cos\theta}{\sin^2\theta}, \quad \dot{\varphi} = \frac{A}{C}b - \frac{(a - b\cos\theta)\cos\theta}{\sin^2\theta} \tag{1.5}$$

The momentum $p_\theta$, corresponding to the positional coordinate $\theta$, depends on the motion of the point $O$ and given by the equation

$$p_\theta = A\dot{\theta} - mz_G\dot{\xi}\sin\theta \tag{1.6}$$

From (1.2), (1.3), (1.5) and (1.6) we obtain the following expression for the Hamilton function (unimportant terms which are functions of time or are constant are omitted)

$$H = \frac{A(a - b\cos\theta)^2}{2\sin^2\theta} + \frac{(p_\theta + mz_G\dot{\xi}\sin\theta)^2}{2A} + mgz_G\cos\theta \tag{1.7}$$

The Hamiltonian (1.7) corresponds to a system with one degree of freedom with generalized coordinate $\theta$.

We will further assume that $\xi(t) = a_*\cos\Omega t$. We will introduce the dimensionless time $\tau = \Omega t$ and dimensionless parameters of the problem and the momentum $p_\theta$ by the formulae $a = \Omega a'$, $b = \Omega b'$, $p_\theta = A\Omega p_\theta'$. Hamiltonian (1.7) can then be rewritten in the form

$$H' = \frac{(a' - b'\cos\theta)^2}{2\sin^2\theta} + \frac{1}{2}(p_\theta' - c\sin\tau\sin\theta)^2 + d\cos\theta, \tag{1.8}$$

where

$$c = \frac{mz_G a_*}{A}, \quad d = \frac{mgz_G}{A\Omega^2} \quad (c > 0, \ d > 0)$$

We will further assume that: (1) the amplitude $a_*$ of the vibrations of the point $O$ is small compared with the characteristic dimension of the body, (2) the natural frequency $\sqrt{(g/l)}$ ($l = A/(mz_G)$ is the reduced length) of small oscillations of the body as a physical pendulum (when $a' = b' = 0$) in the neighbourhood of stable equilibrium $\theta = \pi$ is much less than the frequency $\Omega$ of the vibrations of the point $O$, and (3) the quantities $a'$ and $b'$, representing the angular velocities of natural rotation of the body $\varphi$ and the rotation of its axis of symmetry around the vertical $\psi$, are small. Taking these assumptions into account, we have that

$$c = a_*/l = \varepsilon^2 \ (0 < \varepsilon \ll 1), \quad d = g/(\Omega^2 l) = \varepsilon^4\gamma \ (\gamma > 0), \quad a' = \varepsilon^2\alpha, \quad b' = \varepsilon^2\beta$$

The parameters $\alpha$ and $\beta$ can be taken to be arbitrary quantities. We will further assume that

$\alpha^2 \neq \beta^2$. The case $\alpha^2 = \beta^2$, when the axis of symmetry of the top can occupy the vertical position ($\theta = 0$ or $\theta = \pi$) requires a special consideration.

Making the change of variables $\theta, p'_\theta \to x, X$ in the Hamiltonian (1.8) using the formulae $\theta = x$, $p'_\theta \to \varepsilon X$, we can rewrite it, taking the notation employed into account, in the form

$$H' = H_0 + \varepsilon H_1 + \frac{1}{2!}\varepsilon^2 H_2 + \frac{1}{3!}\varepsilon^3 H_3 \tag{1.9}$$

$$H_0 = 0, \quad H_1 = \frac{1}{2}X^2, \quad H_2 = -2X\sin\tau\sin x$$

$$H_3 = 3\left[\sin^2\tau\sin^2 x + 2\gamma\cos x + \frac{(\alpha - \beta\cos x)^2}{\sin^2 x}\right]$$

We will further carry out the canonical transformation $x, X \to q, p$, $2\pi$-periodic with respect to $\tau$, such that the new Hamilton function does not contain the time $\tau$ in terms up to the third order inclusive in $\varepsilon$. We obtain its transformation using the Depry–Hori method [10].

The new Hamiltonian $K(q, p, \tau)$ must have the following structure

$$K = K_0 + \varepsilon K_1 + \frac{1}{2!}\varepsilon^2 K_2 + \frac{1}{3!}\varepsilon^3 K_3 + O(\varepsilon^4) \tag{1.10}$$

where $K_0 = 0$ and the functions $K_1$, $K_2$ and $K_3$ are found from the relation [10]

$$K_1 = H_1 - \partial W_1/\partial t, \quad K_2 = H_2 + L_1 H_1 + K_{1,1} - \partial W_2/\partial t$$

$$K_3 = H_3 + L_1 H_2 + 2L_2 H_1 + 2K_{1,2} + K_{2,1} - \partial W_3/\partial t$$

Here $L_j = (f, W_j)$ is the Poisson bracket of the functions $f$ and $W_j$, $K_{1,1} = L_1K_1$, $K_{1,2} = L_1K_2$, $K_2, 1 = L_2K_1 - L_1K_{1,1}$, while the functions $W_i(q, p, \tau)$ ($i = 1, 2, 3$) are chosen so that the quantities $K_i$ ($i = 1, 2, 3$) do not contain $\tau$. Calculations show that

$$W_1 = 0, \quad W_2 = 2p\cos\tau\sin q, \quad W_3 = -\frac{3}{4}\sin 2\tau\sin^2 q - 6p^2\sin\tau\cos q.$$

$$K_1 = \frac{1}{2}p^2, \quad K_2 = 0, \quad K_3 = \frac{3}{2}\sin^2 q + 6\gamma\cos q + \frac{3(\alpha - \beta\cos q)^2}{\sin^2 q} \tag{1.11}$$

Simultaneously with the transformation of the Hamiltonian, we shall seek a corresponding canonical replacement of variables having the form

$$x = q + \varepsilon q^{(1)} + \frac{1}{2!}\varepsilon^2 q^{(2)} + \frac{1}{3!}\varepsilon^{(3)} q^{(3)} + O(\varepsilon^4)$$

$$X = p + \varepsilon p^{(1)} + \frac{1}{2!}\varepsilon^3 p^{(2)} + \frac{1}{3!}\varepsilon^2 p^{(3)} + O(\varepsilon^4)$$

The functions $q^{(i)}(q, p, \tau)$ and $p^{(i)}(q, p, \tau)$ ($i = 1, 2, 3$) are obtained using the formulae of the Depry–Hori method [10] (which are not derived here) using the expressions for $W_i$ ($i = 1, 2, 3$) from (1.11). These functions have the form

$$q^{(1)} = 0, \quad q^{(2)} = 2\cos\tau\sin q, \quad q^{(3)} = -12p\sin\tau\cos q$$

$$p^{(1)} = 0, \quad p^{(2)} = -2p\cos\tau\cos q, \quad p^{(3)} = -0.75\sin 2\tau\sin 2q - 6p^2\sin\tau\sin q$$

After substituting the functions $K_i$ from (1.11) into (1.10) we make one more canonical univalent replacement of variables $q, p \to u, v$, given by the formulae $u = \cos q, p = -v\sin q$, which reduce (1.1) to algebraic form. We have

$$K = \varepsilon v^2(1 - u^2)/2 + \varepsilon^3\Pi(u) + O(\varepsilon^4)$$

$$\Pi(u) = \frac{1}{4}(1 - u^2) + \gamma u + \frac{(\alpha - \beta u)^2}{2(1 - u^2)} \tag{1.12}$$

The equations of motion corresponding to (1.12) have the form

$$\frac{du}{d\tau}=\frac{\partial K}{\partial v},\quad \frac{dv}{d\tau}=-\frac{\partial K}{\partial u} \tag{1.13}$$

## 2. THE APPROXIMATE SYSTEM AND ITS EQUILIBRIUM POSITIONS

If we neglect terms $O(\varepsilon^4)$, the following autonomous system of differential equations will correspond to the truncated Hamiltonian obtained

$$\frac{du}{d\tau}=\varepsilon v(1-u^2),\quad \frac{dv}{d\tau}=\varepsilon uv^2-\varepsilon^3\frac{d\Pi}{du} \tag{2.1}$$

We will obtain the equilibrium positions of approximate system (2.1). Since $u\neq\pm1$, in the equilibrium position $v=0$, and the quantity $u$ satisfies the relation $d\Pi/du=0$, where $d\Pi/du=f(u)-u/2+\gamma$ and $f(u)=(\alpha-\beta u)(\alpha u-\beta)(1-u^2)^{-2}$.

The equation

$$f(u)=\frac{1}{2}u-\gamma \tag{2.2}$$

which the equilibrium values of $u$ satisfy, will be investigated graphically in the interval $(-1, 1)$; its roots will be the abscissae of the points of intersection of the curve $y=f(u)$ and the straight line $y=u/2-\gamma$.

As an analysis shows, the function $y=f(u)$ increases monotonically in the interval $(-1, 1)$ for any admissible values of $\alpha$ and $\beta$. Its derivative

$$f'(u)=\frac{(\alpha^2+\beta^2)(1+3u^2)-2\alpha\beta u(3+u^2)}{(1-u^2)^3} \tag{2.3}$$

has a minimum at $u=u_*$, which is the root of the equation $f'(u)=0$, where

$$f''(u)=\frac{6[2(\alpha^2+\beta^2)u(1+u^2)-\alpha\beta(1+6u^2+u^4)]}{(1-u^2)^4}$$

where $u_*\geqslant 0$ when $\alpha\beta\geqslant 0$ and $u_*<0$ when $\alpha\beta<0$. When $-1<u<u_*$ the function $y=f'(u)$ decreases monotonically, and when $u_*<u<1$ it increases monotonically. Graphs of the functions $y=f(u)$ and $y=f'(u)$ are shown in Fig. 1 for the case when $\alpha\beta>0$. The curve $y=f(u)$ intersects the ordinate axis at the point $(0, -\alpha\beta)$, while the curve $y=f'(u)$ intersects the ordinate axis at the point $(0, \alpha^2+\beta^2)$.

The following cases of the intersection of the curve $y=f(u)$ and the straight line $y=u/2-\gamma$ are possible.

*Case* 1. If $f'(u)>1/2$ for all $u\in(-1, 1)$ (Fig. 1a), the curve $y=f(u)$ at each point will be "steeper" than the straight line $y=u/2-\gamma$, which has a constant slope. For any value of the parameter $\gamma(\gamma>0)$ the graphs of the functions $y=f(u)$ and $y=u/2-\gamma$ intersect at a single point, the abscissa of which will henceforth be denoted by $\hat{u}$, and system (2.1) has a unique equilibrium position.

*Case* 2. If $f'(u^*)<1/2$, the straight line $y=1/2$ intersects the graph of the function $y=f'(u)$ at two points (Fig. 1b) and the equation $f'(u)=1/2$ has two solutions, which we will denote by $u_{(1)}$ and $u_{(2)}$ $(u_{(1)}<u_{(2)})$. At points with abscissae $u=u_{(i)}$ $(i=1, 2)$ the straight lines $y=u/2-\gamma_{(i)}$, shown in Fig. 1(b) by the dash-dot lines, touch the curve $y=f(u)$; the quantities $\gamma_{(i)}$ $(i=1, 2)$ are functions of the parameters $\alpha$ and $\beta$.

If $\gamma_{(1)}>0$ and $\gamma_{(2)}>0$, then for values of the parameter $\gamma$ from the intervals $0<\gamma<\gamma_{(1)}$ and $\gamma>\gamma_{(2)}$ the graphs of the functions $y=f(u)$ and $y=u/2-\gamma$ intersect at a single point; we will denote its abscissa by $\hat{u}$. When $\gamma_{(1)}<\gamma<\gamma_{(2)}$ the graphs intersect at three points with abscissae $u=u_i$ $(i=1, 2, 3)$, where $u_1<u_{(1)}<u_2<u_{(2)}<u_3$. In this case system (2.1) has one and three equilibrium positions respectively. If $\gamma=\gamma_{(1)}$ or $\gamma=\gamma_{(2)}$, the system has two equilibrium positions.

When $\gamma_{(1)}<0$, $\gamma_{(2)}>0$ we have three equilibrium positions if $0<\gamma<\gamma_{(2)}$ and one equilibrium position if $\gamma>\gamma_{(2)}$; if $\gamma_{(1)}<0$ and $\gamma_{(2)}<0$, then, for all $\gamma>0$ the system has one equilibrium position.

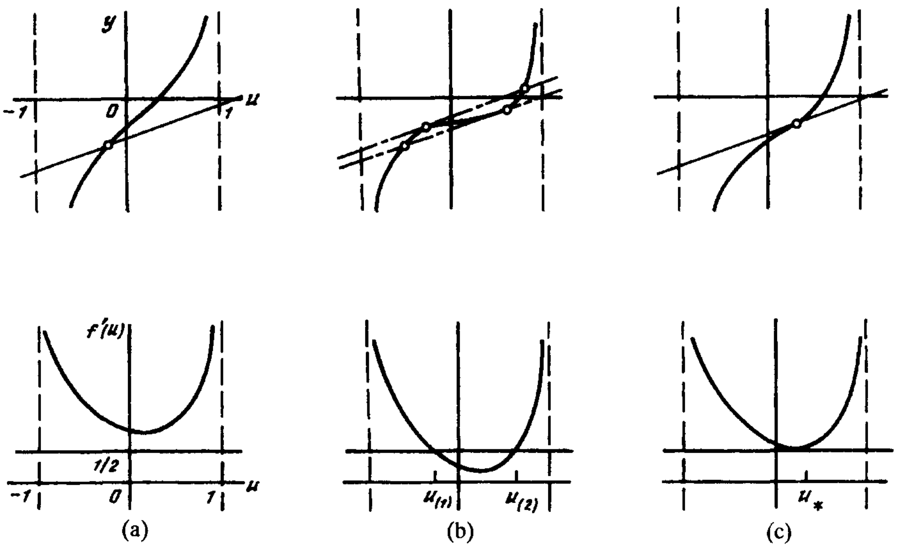



Fig. 1.

The boundary for these two cases is the situation when $f'(u_*) = 1/2$, in which case $u_{(1)} = u_{(2)} = u_*$ and $\gamma_{(1)} = \gamma_{(2)} = \gamma_* = u_*/2 - f(u_*)$. The straight line $y = u/2 - \gamma_*$ touches the curve $y = f(u)$ at its points of inflection (at which $u = u^*$), and system (2.1) has one equilibrium position for all values of $\gamma > 0$ (Fig. 1c).

We will obtain the geometrical position of points in the plane of the parameters $(\alpha, \beta)$, where this boundary situation occurs. The conditions $f'(u) = 1/2$ and $f''(u) = 0$ must be simultaneously satisfied, or, which is equivalent

$$(1-u^2)^3 - 2(\alpha^2+\beta^2)(1+3u^2) + 4\alpha\beta u(3+u^2) = 0$$

$$\alpha\beta(u^4+6u^2+1) - 2(\alpha^2+\beta^2)u(1+u^2) = 0 \tag{2.4}$$

The problem has been reduced to determining those values of the parameters $\alpha$ and $\beta$ for which the two polynomials (2.4) have common roots in the interval $(-1, 1)$. In order that two polynomials should have common roots, it is necessary and sufficient that their resultant $R(\alpha, \beta)$ should be equal to zero [11]. Calculations show that

$$R(\alpha,\beta) = 256(\alpha^2-\beta^2)^6[8\alpha^2-(\beta^2+2)^2][(\alpha^2+2)^2-8\beta^2]$$

Since, by our assumption, $\alpha \neq \pm\beta$, we have $R(\alpha, \beta) = 0$ when

$$\alpha = \pm\frac{\beta^2+2}{2\sqrt{2}} \quad \text{or} \quad \beta = \pm\frac{\alpha^2+2}{2\sqrt{2}} \tag{2.5}$$

In the plane of the parameters $(\alpha, \beta)$ parabolas, having vertices at the points $(\pm\sqrt{(2)}/2, 0)$ and $(0, \pm\sqrt{(2)}/2)$ and which touch one another at the points $(\sqrt{2}, \pm\sqrt{2})$, $(-\pm\sqrt{2})$, correspond to relations (2.5). An analysis shows that for points $(\alpha, \beta)$ of parabolas (2.5) when $|\alpha| \geqslant \sqrt{2}$, $|\beta| \geqslant \sqrt{2}$, the common roots of polynomials (2.4) lie outside the interval $(-1, 1)$, while when $|\alpha| < \sqrt{2}$, $|\beta| < \sqrt{2}$ they lie inside it.

Hence, the transition from Case 1 described above (when system (2.1) has a single equilibrium position for all $\gamma > 0$) to Case 2 (one or three equilibrium positions depending on $\gamma$) and vice versa only occurs on passing through the boundary curve, which is a curvilinear square, the sides of which are parts of parabolas (2.5) when $|\alpha| < \sqrt{2}$, $|\beta| < \sqrt{2}$, with the exception of its vertices, for which $|\alpha| = |\beta| = \sqrt{2}$ (Fig. 2).

Analysis shows that Case 1 occurs outside this square. In this case, if $\alpha\beta < 0$, then $\hat{u} < 0$ (the axis of symmetry of the top makes an obtuse angle with the vertical $OZ$ and the centre of mass of the body is situated below the suspension point $O$); if $\alpha\beta \geqslant 0$, then $\hat{u} > 0$ (the centre of mass lies above the suspension point) when $0 < \gamma < \alpha\beta$ and $\hat{u} \leqslant 0$ when $\gamma \geqslant \alpha\beta$.

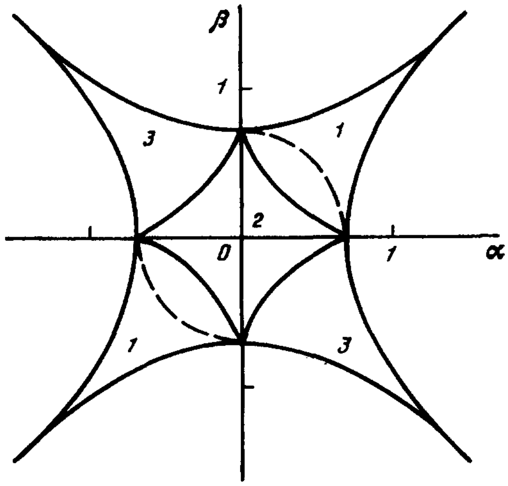



Fig. 2.

Inside the curvilinear square with sides of the parabolas (2.5) we have Case 2. We distinguish in this region the set of points $(\alpha, \beta)$ for which the quantities $\gamma_{(i)}$ $(i = 1, 2)$ vanish. For these points the conditions $f(u) = u/2$ and $f'(u) = 1/2$ must be simultaneously satisfied. This is equivalent to the resultant $G(\alpha, \beta)$ of the polynomial

$$(1 - u^2)^2 u + 2(\alpha - \beta u)(\beta - \alpha u)$$

and the first of polynomials (2.4) is being equal to zero. Calculations show that

$$G(\alpha,\beta) = -256(\alpha^2 - \beta^2)^4 g(\alpha,\beta)$$

$$g(\alpha,\beta) = 128(\alpha^6 + \beta^6) + 27\alpha^4\beta^4 - 48\alpha^2\beta^2(\alpha^2 + \beta^2) -$$
$$-192(\alpha^4 + \beta^4) + 696\alpha^2\beta^2 + 96(\alpha^2 + \beta^2) - 16$$

Since $\alpha \neq \pm\beta$, we have $G(\alpha, \beta) = 0$ when $g(\alpha, \beta) = 0$.

The graph of the curve $g(\alpha, \beta) = 0$ (Fig. 2) is a curvilinear square, the vertices of which $(\pm\sqrt{2}/2, 0)$, $(0, \pm\sqrt{2}/2$ lie on the sides of the curvilinear square with the sides of the parabolas (2.5) (which are the boundaries of the region considered). These vertices are cuspidal points for the curve $g(\alpha, \beta) = 0$; the middle of the "sides" (where $\alpha = \pm\beta$) have coordinates $\alpha \neq \pm\beta$.

On the part of the curve $g(\alpha, \beta) = 0$, which lies in the first and third quadrants of the $(\alpha, \beta)$ coordinate plane, where $\alpha\beta > 0$, $y_{(1)}$ vanishes; on the other part of the curve, in the second and third quadrants, where $\alpha\beta < 0$, $\gamma_{(2)}$ vanishes. In regions 1 and Fig. 2 we have $\gamma_{(1)} > 0$, $\gamma_{(2)} > 0$, in region 2 $\gamma_{(1)} < 0$, $\gamma_{(2)} > 0$, and in regions 3 $\gamma_{(1)} < 0$ and $\gamma_{(2)} < 0$; an analysis of the number of equilibrium positions of system (2.1) as a function of the value of $\gamma$ for each of these regions is carried out in the same way as indicated above.

In regions 3 (a unique equilibrium position for all $\gamma > 0$) we have $\hat{u} < 0$.

In regions 1 and 2 for those values of $\gamma$ for which one equilibrium position exists, we have $\hat{u} > 0$ when $0 < \gamma < \gamma_{(1)}$ and $\hat{u} < 0$ when $\gamma > \gamma_{(2)}$ in regions 1 and $\hat{u} < 0$ when $\gamma > \gamma_{(2)}$ in region 2.

In the case of three equilibrium positions $u = u_i$ $(i = 1, 2, 3)$ in regions 1 when $\alpha^2 + \beta^2 > 1/2$ we have $u_2 > 0$, $u_3 > 0$, and $u_1 \geqslant 0$ when $\gamma_{(1)} < \gamma \leqslant \alpha\beta$ and $u_1 < 0$ when $\alpha\beta < \gamma < \gamma_{(2)}$; on the circle $\alpha^2 + \beta^2 = 1/2$ we have $u_1 < 0 < u_2 < u_3$; for points inside this circle $u_1 < 0$, $u_3 > 0$ and $u_2 < 0$ when $\gamma_{(1)} < \gamma < \alpha\beta$ and $\gamma_{(1)} < \gamma < \alpha\beta$ when $u_2 \geqslant 0$ (the arcs of the circle $\alpha\beta \leqslant \gamma < \gamma_{(2)}$ are shown by the dashed curves in Fig. 2).

In the case of three equilibrium positions in regions 2 we have $u_1 < 0$ and $u_3 > 0$, and if $\alpha\beta \leqslant 0$ then $u_2 > 0$, while if $\alpha\beta > 0$ then $u_2 < 0$ when $0 < \gamma < \alpha\beta$ and $u_2 \geqslant 0$ when $\alpha\beta \leqslant \gamma < \gamma_{(2)}$.

We will now consider the case $\gamma = \gamma_{(1)}$ or $\gamma = \gamma_{(2)}$ in regions 1 and 2, when system (2.1) has two equilibrium positions. If $\gamma = \gamma_{(1)}$ (in regions 1), we have $u_1 = u_2 = u_{(1)} < u_{(2)} < u_3$; if $\gamma = \gamma_{(2)}$ (in regions 1 and 2), then $u_1 < u_1 < u_{(2)} = u_{(3)} = u_{(2)}$. Outside the circle $\alpha^2 + \beta^2 = 1/2$ in these regions we have

$u_{(1)} > 0, u_{(2)} > 0$, and on the circle $u_{(1)} = 0, u_{(2)} \geqslant 0$ when $\alpha\beta \geqslant 0$ and $u_{(1)} > 0, u_{(2)} = 0$ when $\alpha\beta < 0$; inside the circle $u_{(1)} < 0, u_{(2)} > 0$. The signs of non-multiple equilibrium values of $u$ are determined in the same way as in he case of three equilibrium positions.

For points $(\alpha, \beta)$ which belong to the curvilinear square with sides-parabolas (2.5) (the boundary situation when, for all $\gamma > 0$, system (2.1) has one equilibrium position) we have $\hat{u} < 0$ when $\alpha\beta < 0$ and when $\alpha\beta \geqslant 0$ $\hat{u} > 0$, if $0 < \gamma < \alpha\beta$ and $\hat{u} \leqslant 0$, if $\gamma \geqslant \alpha\beta$.

For points of the curvilinear square, described by the equation $g(\alpha, \beta) = 0$ when $\alpha\beta < 0$ we have $\hat{u} < 0$, and when $\alpha\beta \geqslant 0$, $u_1 < 0$, $u_3 > 0$, in this case $u_2 < 0$, if $0 < \gamma < \alpha\beta$ and $u_2 \geqslant 0$ when $\alpha\beta \leqslant \gamma < \gamma_{(2)}$.

## 3. PERIODIC MOTIONS OF THE REDUCED SYSTEM

Motions of a Lagrange top close to regular precessions. We will now consider the motion of the reduced system described by Hamiltonian (1.12).

In the neighbourhood of the equilibrium position $u = \text{const}$, $v = 0$ of approximate system (2.1), the reduced system (1.13) can be regarded as quasilinear with perturbations of the order of $\varepsilon^4$, $2\pi$-periodic in $\tau$. The roots of the characteristic equation, linearized in the neighbourhood of the equilibrium position of the approximate system are of the order of $\varepsilon^2$, and hence, for sufficiently small $\varepsilon$, we have the non-resonance case of Poincaré's theory in the problem of periodic motions [12]. We will eliminate the case $\gamma = \gamma_*$ from consideration for the points $(\alpha, \beta)$, which belong to the boundary of the curvilinear square with sides (2.5), and also the case $\gamma = \gamma_{(1)}$ in regions 1 and $\gamma = \gamma_{(2)}$ in regions 1 and 2 in Fig. 2 (see Section 2). For all the remaining permissible values of the parameters $\alpha$, $\beta$ and $\gamma$ a unique solution of the system with complete Hamiltonian (1.12), analytic in $\varepsilon$ and $2\pi$-periodic in $\tau$, is produced from each equilibrium position of the approximate system.

By making the replacements of variables, described in Section 1, in the reverse order, we obtain that $p_\theta = O(\varepsilon^4)$, and the angle of deviation of the axis of symmetry of the top from the upper vertical position is given by the expression

$$\theta = \arccos u_0 + \varepsilon^2 \sqrt{1-u_0^2}\cos\tau + O(\varepsilon^4) \tag{3.1}$$

where $u_0$ is the equilibrium value of $u$ of the approximate system ($u_0 = \hat{u}$ in the case of one equilibrium position and $u_0 = u_i$ ($i = 1, 2$, or 3) in the case of three equilibrium positions). The quantities $O(\varepsilon^4)$ in the expression for $p_\theta$ and in (3.1) are $2\pi$-periodic in $\tau$.

Using relations (1.5) and the relation between the constants $a, b$ and $\alpha, \beta$ (see Section 1), we obtain the following expressions for the angular velocities of precession and natural rotation

$$\frac{d\psi}{d\tau} = \varepsilon^2 \frac{\alpha - \beta\cos\theta}{\sin^2\theta}, \quad \frac{d\varphi}{d\tau} = \varepsilon^2\left[\frac{A}{C}\beta - \frac{(\alpha - \beta\cos\theta)\cos\theta}{\sin^2\theta}\right] \tag{3.2}$$

The quantity $\theta$ is defined in (3.1).

Relations (3.1) and (3.2) give the motions of the Lagrange top with a vibrating point $O$, close to regular precession: the nutation angle $\theta$ differs from a constant by a quantity of the order of $\varepsilon^2$, while the angular velocities $d\psi/d\tau$, $d\varphi/d\tau$, being small quantities of the order of $\varepsilon^2$, differ from constants by quantities of the order of $\varepsilon^4$; these "corrections" are $2\pi$-periodic in $\tau$.

Note that in the case of a Lagrange top with a fixed point there is only one value of the angle $\theta$ for each pair $(a, b)$ of parameters of the problem ($a \neq \pm b$), for which a regular precession of the top exists. When the point $O$ vibrates, as considered in this paper, there will be one or three motions, close to regular precession, depending on the value of the parameters $\alpha$, $\beta$ and $\gamma$.

## 4. THE STABILITY OF THE MOTIONS OF A TOP, CLOSE TO REGULAR PRECESSIONS

We will consider the stability of the motions of a Lagrange top, described in Section 3, with a vibrating point $O$, close to regular precessions, with respect to the variables $\theta$ and $p_\theta$. Solutions of the reduced system (1.13), $2\pi$-periodic in $\theta$, of the form $u^* = u_0 + O(\varepsilon^4)$, $\upsilon^* = O(\varepsilon^4)$ correspond to these motions of the top.

Assuming $u = u^* + x$, $\upsilon = \upsilon^* + y$, we will write the Hamiltonian of the perturbed motion in the form of a series in powers of $x$ and $y$

$$\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4 + \dots \tag{4.1}$$

where $\Gamma_k$ is the form of the $k$th power with respect to $x$ and $y$ with coefficients which are functions, $2\pi$-periodic in $\tau$, if we neglect terms of the order of $\varepsilon^4$ and higher, we have

$$\Gamma_2 = \frac{1}{2}\varepsilon(1-u_0^2)y^2 + \frac{1}{2}\varepsilon^3\left[f'(u_0) - \frac{1}{2}\right]x^2$$

$$\Gamma_3 = -\varepsilon u_0 xy^2 + \frac{1}{6}\varepsilon^3 f''(u_0)x^3$$

$$\Gamma_4 = -\frac{1}{2}\varepsilon x^2 y^2 + \frac{1}{24}\varepsilon^3 f'''(u_0)x^4$$

In the linear approximation the question of the stability of the motions considered is determined by the sign of the coefficient $f'(u_0) - 1/2$ of the form $\Gamma_2$: if $f'(u_0) > 1/2$ we have stability and if $f'(u_0) < 1/2$ we have instability.

The sign of the quantity $f'(u_0) - 1/2$ for the case of one and three equilibrium positions $u = u_0$, $\upsilon = 0$ of the approximate system, which produce the periodic motions considered, can be obtained from the results in Section 2. In the case of a single equilibrium position $u_0 = \hat{u}$, $\upsilon = 0$ we have $f'(u) > 1/2$ (see Fig. 1a), and the corresponding periodic solution is stable in the linear approximation. In the case of three equilibrium positions, when $u_0 = u_i$ ($i = 1, 2, 3$), taking into account the fact that $u_1 < u_{(1)} < u_2 < u_{(2)} < u_3$ and also that $f'(u_0) > 1/2$ when $-1 < u < u_{(1)}$ and $u_{(2)} < u < 1$ and $f'(u) < 1/2$ when $u_{(1)} < u < u_{(2)}$ (Fig. 1b), we obtain that the periodic motions corresponding to the larger equilibrium value of $u$ ($u_3$) and the smaller equilibrium value of $u$ ($u_1$) are stable, and the motion corresponding to the middle equilibrium value of $u$ ($u_2$) is unstable in the linear approximation.

This periodic motion, which is unstable in the linear approximation, remains unstable in the non-linear problem also, as follows from the Lyapunov's theorem on stability in the first approximation [13].

We will give a rigorous solution of the problem of the stability of the periodic motions of the system in question, which are stable in the linear approximation. To solve it we will carry out a non-linear analysis using the results of the KAM-theory [14].

Using a Birkhoff canonical transformation $x, y \to X, Y$, we can reduce the Hamilton function (4.1) to normal form [15]

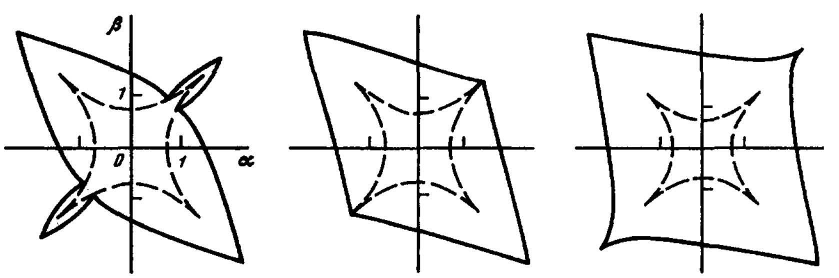
$$\Gamma = \frac{1}{2}\omega(X^2 + Y^2) + \frac{1}{4}c(X^2 + Y^2)^2 + O_5$$

where $O_5$ is a set of terms, $2\pi$-periodic in $\tau$, the power of which with respect to $X$ and $Y$ is no less than five, and $\omega$ and $c$ are constant coefficients which, as calculations show, have the following form

$$\omega = \varepsilon^2\sqrt{(1-u_0^2)(f'(u_0) - ½)} + O(\varepsilon^4)$$

$$c = -\frac{\varepsilon c_1}{16(f'(u_0) - ½)^2(1-u_0^2)^6} + O(\varepsilon^2)$$

$$c_1 = 4(3u_0^4 + 15u_0^2 - 2)(\alpha^4 + \beta^4) - 8u_0(3u_0^4 + 20u_0^2 + 9)\alpha\beta(\alpha^2 + \beta^2) +$$




(a) (b) (c)

Fig. 3.

$$+4(u_0^6+39u_0^4+45u_0^2+11)\alpha^2\beta^2+2(15u_0^4+32u_0^2+1)(1-u_0^2)^2(\alpha^2+\beta^2)-$$
$$-8u_0(u_0^4+17u_0^2+6)(1-u_0^2)^2\alpha\beta+(2u_0^2+1)(1-u_0^2)^5$$

If $c \neq 0$, then, by the Arnol'd–Moser theorem [14], the periodic motion under consideration is stable. For fairly small $\varepsilon$ the stability condition will break down when the equality $c_1 = 0$ is satisfied, where the quantity $u_0$ occurring in $c_1$ is the root of Eq. (2.2).

To investigate the system formed by Eqs (2.2) and $c_1 = 0$, we will introduce the new parameters

$$\sigma_1 = \alpha^2 + \beta^2, \quad \sigma_2 = \alpha\beta \; (\sigma_1 > 0) \tag{4.2}$$

The quantities $\sigma_1$ and $\sigma_2$ must satisfy the obvious inequalities

$$\sigma_1 + 2\sigma_2 > 0, \quad \sigma_1 - 2\sigma_2 > 0 \tag{4.3}$$

where an equality sign in (4.3) is impossible, since by our assumption $\alpha \neq \pm\beta$.

The equation $c_1 = 0$ is quadratic, while Eq. (2.2) is linear in $\sigma_1$ and $\sigma_2$; they have the form

$$\begin{gathered}4(3u_0^4+15u_0^2-2)\sigma_1^2+4(u_0^6+33u_0^4+15u_0^2+15)\sigma_2^2-\\ -8u_0(3u_0^4+20u_0^2+9)\sigma_1\sigma_2+2(15u_0^4+32u_0^2+1)(1-u_0^2)^2\sigma_1-\\ -8u_0(u_0^4+17u_0^2+6)(1-u_0^2)^2\sigma_2+(2u_0^2+1)(1-u_0^2)^5=0\\ u_0\sigma_1-(1+u_0^2)\sigma_2=(u_0/2-\gamma)(1-u_0^2)^2\end{gathered} \tag{4.4}$$

Expressing $\sigma_2$ in terms of $\sigma_1$ from the second equation of (4.4) and substituting it into the first, we obtain the equation

$$\begin{gathered}8\sigma_1^2-2[15u_0^4+1+8u_0\gamma(3-u_0^2)]\sigma_1-[4(u_0^6+33u_0^4+15u_0^2+15)\gamma^2-\\ -4u_0(3u_0^6+69u_0^4+61u_0^2+27)\gamma+(3u_0^8+102u_0^6+108u_0^4+42u_0^2+1)]=0\end{gathered} \tag{4.5}$$

We will consider it as quadratic with respect to $\sigma_1$. We will be interested in the conditions for which the roots of this equation are real, positive and, moreover, satisfy relations which follow from inequalities (4.3). These relations have the form

$$\sigma_1 > (u_0-2\gamma)(1-u_0)^2, \quad \sigma_1 > -(u_0-2\gamma)(1+u_0)^2 \tag{4.6}$$

Equation (4.5) has real solutions if its discriminant

$$D = 12(1+u_0^2)^2 D_1$$
$$D_1 = 32(u_0^2+5)\gamma^2-16(7u_0^2+17)u_0\gamma+(83u_0^4+106u_0^2+3)$$

is non-negative. We will consider $D_1$ as a quadratic trimonial in $\gamma$; its discriminant $-1920(1-u_0^2)^3 < 0$ when $|u_0| < 1$, and hence the condition $D_1 > 0$, and, of course, $D > 0$ is satisfied for all permissible values of the parameters $\gamma$ and $u_0$. Hence, Eq. (4.5) has two roots, equal to

$$[15u_0^4+1+8u_0\gamma(3-u_0^2)\pm(1+u_0^2)\sqrt{3D_1}\,]/8 \tag{4.7}$$

An investigation of their signs shows that the larger root (which we will henceforth denote by $\sigma_1^*$) is always positive; the smaller root is either negative or positive, but it then does not satisfy conditions (4.6). The root $\sigma_1^*$ satisfies conditions (4.6) for any admissible values of the parameters $u_0$ and $\gamma$.

Hence, Eq. (4.5) has the single solution $\sigma_1 = \sigma_1^*$ which satisfies all the postulated conditions. The following value of $\sigma_2^*$ corresponds to it

$$\sigma_2 = \sigma_2^* = \frac{\sigma_1^* u_0-(u_0/2-\gamma)(1-u_0^2)^2}{1+u_0^2} \tag{4.8}$$

obtained from the second equation of (4.4).

Returning once again to the parameters $\alpha$ and $\beta$, according to relations (4.2), we obtain four pairs of values of $\alpha$ and $\beta$ corresponding to the pair of values $\sigma_1$ and $\sigma_2$

$$\alpha_{1,2} = (A \pm B)/2, \quad \beta_{1,2} = (A \mp B)/2$$
$$\alpha_{3,4} = (-A \pm B)/2, \quad \beta_{3,4} = (-A \mp B)/2 \tag{4.9}$$
$$A = \sqrt{\sigma_1^* + 2\sigma_2^*}, \quad B = \sqrt{\sigma_1^* - 2\sigma_2^*}$$

Suppose the value of $\gamma$ is fixed. Then relations (4.9), taking expressions (4.7) and (4.8) into account, specify curves in the $(\alpha, \beta)$ plane in parametric form (the parameter is the quantity $u_0$, which varies from $-1$ to 1), where the sufficient condition for stability of the periodic motion of the system considered breaks down.

The form of these curves depends on the value of the parameter $\gamma$. Analysis shows that the case when $0 < \gamma < 1$ and $\gamma > 1$ will be qualitatively different. In Fig. 3(a–c) we show the cases $0 < \gamma < 1$, $\gamma = 1$, $\gamma > 1$, respectively. In all these cases curves (4.9) form a closed contour, shown in Fig. 3 by the continuous curve. The contour has vertices at points with coordinates $(\pm\sqrt{(2(2-\gamma))}, \pm\sqrt{(2(2-\gamma))})$ when $\gamma < 1$ and $(\pm\sqrt{(2(2\gamma-1))}, \pm\sqrt{(2(2\gamma-1))})$ when $\gamma \geqslant 1$, when $\alpha = \beta$ (corresponding to $u_0 = 1$), and also vertices at points with coordinates $(\pm\sqrt{(2(2+\gamma))}, \pm\sqrt{(2(2+\gamma))})$, when $\alpha = -\beta$ (in this case $u_0 = -1$).

The dashed curve in Fig. 3 represents a curvilinear square with sides-parabolas (2.5), inside which, at each point $(\alpha, \beta)$, there is one or three periodic motions of the system (depending on $\gamma$), while outside there is only one such motion. When $\gamma > 1$ the contour of curves (4.9) as a whole contains a curvilinear square and its boundary (Fig. 3c). When $0 < \gamma < 1$ each of curves (4.9) also lies outside the region occupied by this square, but, with one of the parabolas (2.5), there is associated a point which is a cuspidal point for the curve (Fig. 3a); the coordinates of the cuspidal points are $(\pm(+\gamma^2)/\sqrt{2}, \pm\sqrt{(2)}\gamma)$ and $(\pm\sqrt{(2)}\gamma, \pm(1+\gamma^2)/\sqrt{2})$, and the value $u_0 = \gamma$ of the parameter $u_0$ corresponds to these points. When $\gamma = 1$ the cuspidal points of curves (4.9) merge in pairs and coincide with the vertices of the contour formed by these curves (Fig. 3b).

Hence, the $2\pi$-periodic motions of this system, which is stable in the linear approximation and corresponds to points $(\alpha, \beta)$ inside the curvilinear square with sides (2.5), are also stable for any values of $\gamma$ in the non-linear problem. The unique periodic motion for points outside this square, which is stable in the linear approximation, is also stable, with the exception, possibly, of points $(\alpha, \beta)$ lying on curves (4.9).

The conclusions reached regarding the stability of periodic motions of this system can be extended to the corresponding motions of a Lagrange top, close to regular precessions: for points $(\alpha, \beta)$ inside a square with sides (2.5) they are either one stable motion of the top (depending on $\gamma$) or three such motions, two of which are stable and one of which is unstable; outside the square there is a single motion which is stable for all permissible values of $\alpha$, $\beta$ and $\gamma$, with exception, perhaps, of points belonging to curves (4.9).

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